J. Fluid Mech. (1985), vol. 157, pp. 289–303
Printed in Great Britain

Bifurcation phenomena in flows between a rotating circular cylinder and a stationary square outer cylinder

By T. MULLIN AND A. LORENZEN†
Mathematical Institute, 24/29 Gt Giles, Oxford

(Received 9 March 1984 and in revised form 22 February 1985)

The properties of steady cellular flows between a rotating cylinder and a stationary outer one of square cross-section have been investigated experimentally. Results for the primary-flow selection process, which involves four-cell and six-cell states, are intricate but bear a reassuring qualitative resemblance to those obtained previously for the standard Taylor–Couette model. In addition, the existence of anomalous modes disconnected from the primary flow has been demonstrated and their dependence upon the length of the flow domain studied. The phenomena are found to be in accord with the abstract theoretical framework that has been successfully used to interpret previous experimental observations on steady-flow problems with multiple solutions.

1. Introduction

For over sixty years the Taylor–Couette model has been a heavily researched area of hydrodynamic stability. The various flows observable between concentric circular cylinders, the inner of which rotates, are the subject of a large experimental and theoretical literature, and standard conceptions about hydrodynamic stability and the origins of turbulence in confined flows are sometimes focused on this archetype. A precise account of at least those Taylor–Couette flow phenomena observed at moderate Reynolds numbers has been established in recent years, which is more or less complete even though it has turned out to be much more intricate than had previously been supposed. The Taylor–Couette model is nevertheless highly special, in particular because of its azimuthal symmetry which simplifies the theoretical problem but may induce non-generic properties. It has therefore become an objective for further research to inquire into the wider implications of facts that have so far been made secure only in relation to the Taylor–Couette model. Which of the established facts represent universal properties of flow systems and which do not, so needing cautious treatment in relation to general ideas about the origins of turbulence?

The present experimental study is a step in this direction. We have examined a class of flows superficially similar to Taylor–Couette flows but with a gross difference in boundary geometry such that a quantitatively close duplication of properties cannot be expected. Moreover, the major exemplary attribute of the flows here investigated is that they are three-dimensional and too complicated for any hope of constructive analysis. In view of the precision that would be needed to reproduce the

† Present address: Max-Planck-Institut für Strömungsforschung, Böttingerstr. 4–8, D-3400 Göttingen, W. Germany

delicate properties of these flows exposed experimentally, they can be classified inaccessible to present-day computing resources. As evidence supporting general notions about nonlinear hydrodynamic problems, therefore, those properties found to be shared at least qualitatively with the Taylor–Couette model have special interest, just as conversely do certain other properties that are radically different.

To summarize the background to the present investigation it may be recalled that the general theoretical conclusions by Benjamin (1978*a*) concerning bifurcations among steady solutions of the Navier–Stokes equations have been tested in experiments on Taylor–Couette flows by Benjamin (1978*b*), Mullin (1982) and Mullin, Pfister & Lorenzen (1982). Theoretical progress was also made by Schaeffer (1980), who considered the classic theoretical model with axial periodicity conditions to be an 'organizing centre' for perturbed bifurcation phenomena as observed in real Taylor–Couette steady flows, and Schaeffer's model has been explored in detail for a specific case by Hall (1982). Further progress in the understanding of these flows is most likely to come from numerical studies of the exacting kind reported recently by Cliffe (1983). As already explained one of the present aims is to extend the experimental testing of Benjamin's conclusions by examining another class of steady flows, with less-simple boundary conditions, which is beyond the reach of any theoretical treatment except by his abstract qualitative analysis. Some complementary experimental observations, dealing with the effects of breaking geometrical symmetries on the transition to turbulence in Taylor–Couette flows, have already been reported by Mullin, Lorenzen & Pfister (1983).

The experimental system consists of a rotating circular cylinder positioned centrally in a stationary outer one of square cross-section. Thus with this geometry the flow domain possesses only discrete symmetries in the azimuthal direction, in contrast with the standard Taylor–Couette configuration which has continuous symmetry. The stationary top and bottom end plates provide symmetric boundary conditions and determine the length of the flow column. The position of the top plate is continuously adjustable, so that the influence of length as a continuous parameter can be studied.

When the Reynolds number R is sufficiently small there is a unique solution to the Navier–Stokes equations. In classifying all the practically possible states of steady flow for a given set of boundary conditions, we can in general define the *primary flow* as that developed by very gradual increase in R from small values. In the present case this flow consists of an even number of counter-rotating steady vortices superposed upon the main azimuthal motion. The number of these vortices which constitute the primary flow will obviously depend upon the length of the flow domain, and this dependence will be a major subject of our investigation.

The occurrence of steady vortices at moderate values of R may at first seem surprising in view of the complicated recirculating flows present in the corners. However, a numerical study of a two-dimensional version of the problem has been performed by Lewis (1979), whose findings indicate that the recirculations will be weak effects. Indeed, some observations on steady vortices in the present geometry have been reported by Snyder (1968).

Besides the primary flow, several other steady flows are possible for any particular length. These consist of even or odd numbers of vortices and can be produced in practice by sudden acceleration of the inner cylinder to a value of R above some critical range. Evidently they cannot survive when R is gradually reduced below a critical value R_c. Those states with an even number of vortices, which are different from the primary flow but which also have inward flow at the fixed ends, are defined to be *normal* secondary flows. Anomalous modes are defined to be those which have a direction of spiralling of the end cells such that outward flow is found at either one or both endwalls. Thus anomalous modes may consist of either an even or an odd number of vortices, in the latter case being realizable in two configurations with the inward spiralling cell at the top or bottom respectively. A full discussion of the properties of anomalous modes in the standard Taylor–Couette geometry is given in Benjamin & Mullin (1981) and Cliffe & Mullin (1985).

The greater part of this study is concerned with the exchange of priority between the primary four-cell and six-cell flows as the length of the flow domain is changed. This exchange process was in practice exposed by studying the behaviour of the primary flows as a function of R for various fixed values of length. The anomalous four-cell and five-cell flows have also been investigated, and their behaviour as a function of the Reynolds number and length is discussed. In addition to these steady flows a steady asymmetric four-cell state was investigated. Finally a region of parameter space (R, Γ), where Γ is aspect ratio, was found where no steady flow was realizable even although the Reynolds number was relatively small.

2. Apparatus

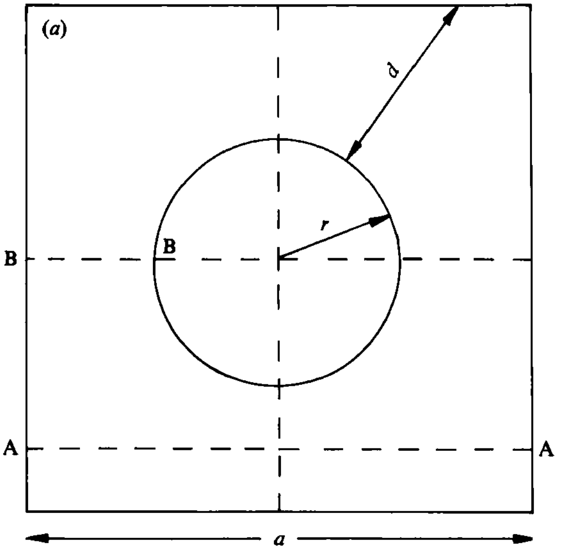
A plan view of the working section is shown in figure 1 (*a*) and an overall view may be seen in the photograph presented as figure 1 (*b*). The inner cylinder is machined from stainless steel and has a diameter of 31.65 ± 0.2 mm. It is supported between PTFE bearings which are enclosed in the thick Perspex base of the apparatus and in the tight-fitting lid. The square cylinder was manufactured from selected glass plates which are located in milled slots in the thick Perspex top and bottom. The square cylinder was glued together on a milled aluminium former which ensured that an accurate square cross-section was produced. The side of the square is $a = 63.58 \pm 0.05$ mm. The gap width d varies from 15.97 mm, in the narrowest part between the inner cylinder and the square box, to 29.14 mm in the corners.

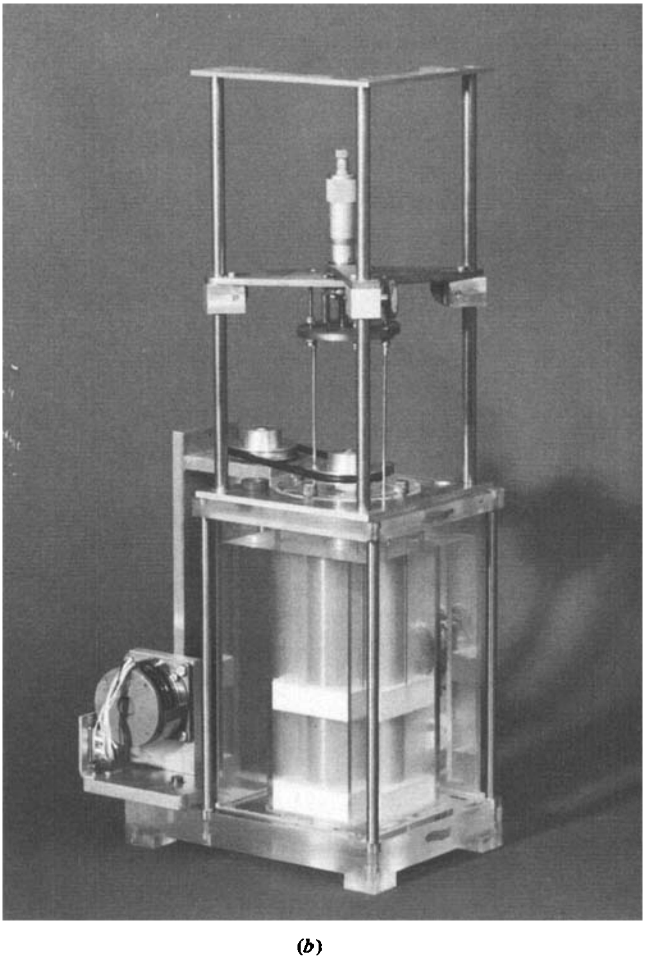
The two white PTFE collars seen in the photographs determine the effective lengths of the fluid column. One is fitted into the bottom of the apparatus and the other is held by two thin rods. These rods are attached to the overhead gantry which facilitates accurate changes in length to be made using a micrometer. Alternatively, the micrometer section of the gantry can be released and repositioned, which adjustment allows for larger changes in length. We shall use the narrow gap width to non-dimensionalize the lengthscale.

The inner cylinder is driven through a 6:1 gear box and a 1:1 pulley system by a stepping motor which operates on 400 steps per revolution. The dimensionless speed is the Reynolds number $R = \omega_i r_i^2/\nu$, where ω_i is the angular frequency, r_i the radius of the inner cylinder and ν the kinematic viscosity of the fluid. The stepping motor is controlled by use of an oscillator whose frequency stability is better than 0.1 %.

The working fluid is an aqueous solution of glycerol approximately 67:33 by volume. A typical value for the viscosity is 4 cSt. A small quantity of pearly substance ('Mearlmaid' AA natural pearl essence) was added for flow visualization. The square cylinder is surrounded by a further glass box and the gap is filled with a clear aqueous glycerol solution so that this fluid jacket acts as a temperature bath. The entire apparatus is placed in an air-temperature-controlled cabinet which ensures the temperature variation in the working fluid to be better than 0.1 K even over experimental runs of one day.

Flows in the annular gap were observed by illuminating with a long narrow slit





(b)

FIGURE 1 (a). Plan view of the working section of the apparatus. (b) Front view of apparatus.

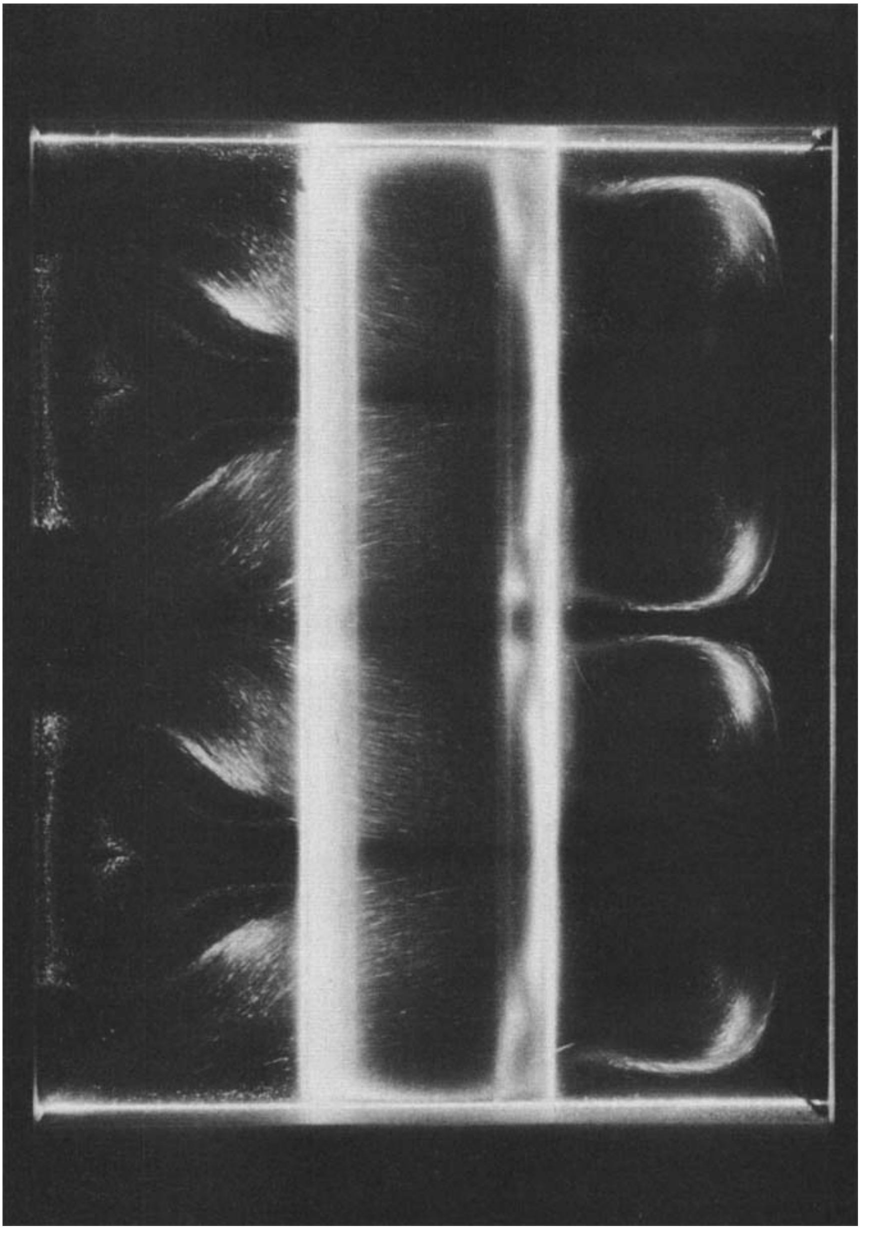


FIGURE 2. Four-vortex state at $R = 131$ and $\Gamma = 5$. The illumination is in the vertical plane AA shown in figure 1 (*a*). The reader is reminded that the main flow is in the azimuthal direction with secondary vortex flow superposed. The flow is towards the reader at the left side and thus the impression of asymmetry is given by artefacts of the method of visualization.

of light which was directed perpendicularly to a wall of the cylinder and in a plane at right angles to the direction of viewing. In this way different sections of the flow could be illuminated to show different facets of the flow, examples of which are given in figures 2 and 3. The cell sizes could readily be measured by use of a cathetometer

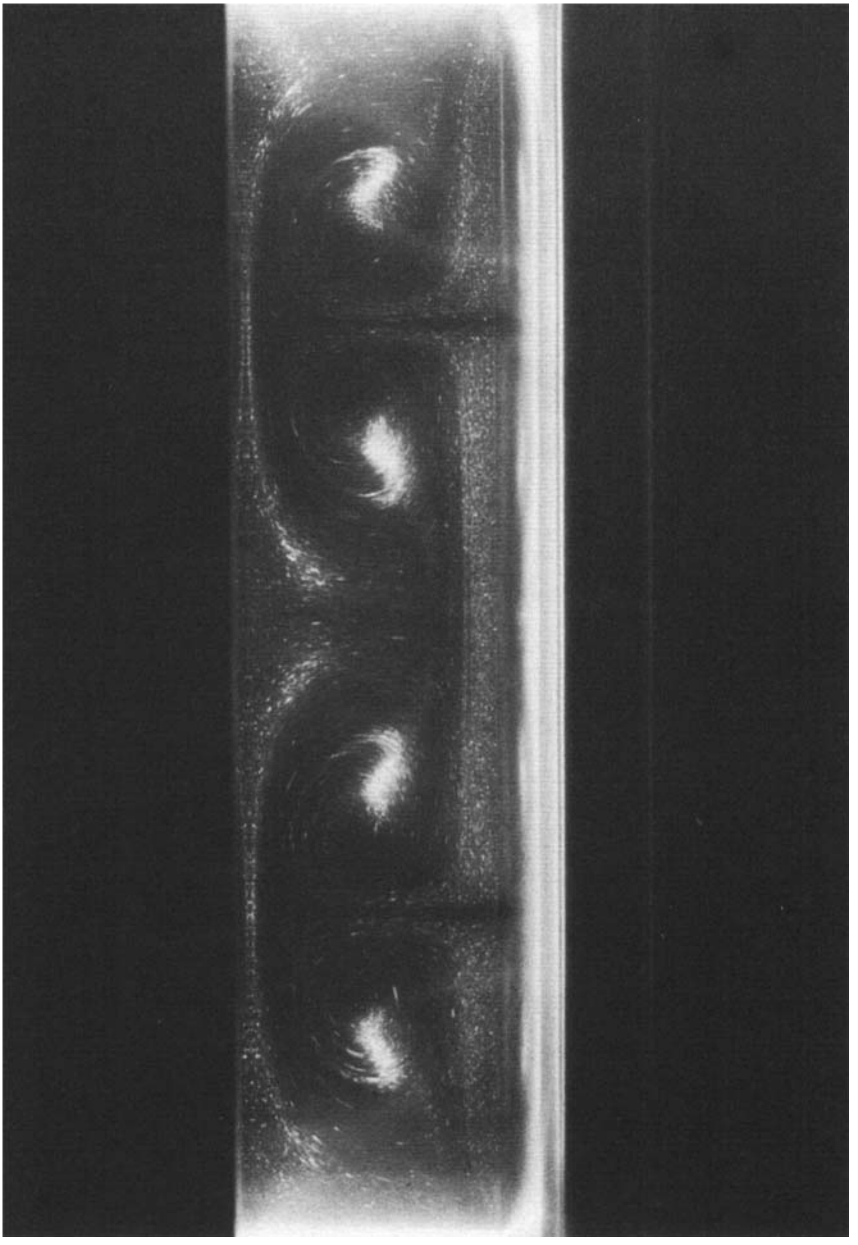


FIGURE 3. Four-vortex state at $R = 131$ and $\Gamma = 5$. The illumination is in the vertical plane BB shown in figure 1 (*a*).

external to the cabinet, which was also used to measure the length of the working section.

The settling times in these experiments were very long. They were apparently determined by the weak re-circulation in the corners and repeatable results could only be obtained after allowing settling times of 20 minutes after 1 % changes in speed.

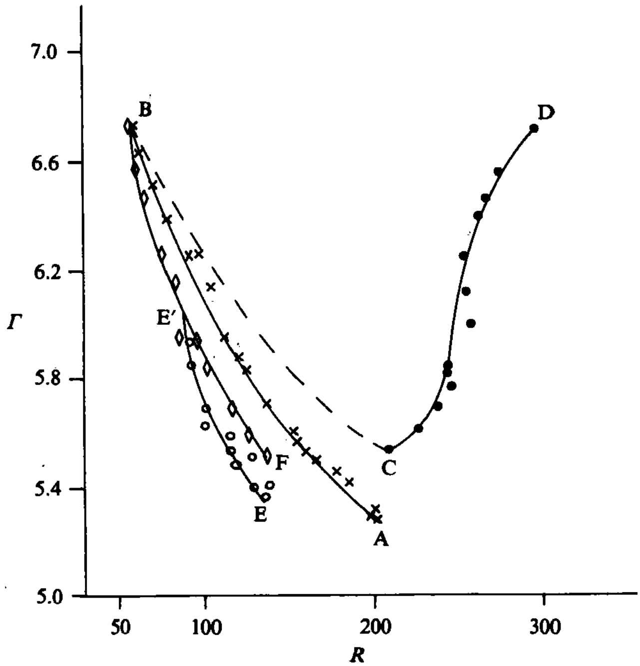


FIGURE 4. Experimentally determined bifurcation set in the (R, Γ)-plane and four and six cells. N.B. BC was not determined experimentally but is drawn in as a dashed line to aid the discussion of the results. AB is the locus of the limits of the six-cell steady flow feature by reducing R. CD is the locus of the limits of stability of the four-cell secondary mode. BF is the locus of bifurcation points for the onset of time-dependent flow by increase of R. E′E is the locus of symmetry-breaking bifurcation points found by increasing R.

3. Results

3.1. *The primary flow exchange*

For small values of R the motion is mainly in circles near the inner cylinder but becomes more distorted from its circular path near the outer square section. Although difficult to observe, a weak recirculation exists in the corners as was shown in the numerical work of Lewis (1979). At the bottom and top boundaries the flow is extremely complicated near the corners. With very gradual increase of R a smooth transition to a vortex flow state takes place. The vortices spread in from the fixed ends until the entire length is filled. The cellular state produced in this way is defined to be the primary flow and consists of an even number of counter-rotating vortices with inward flow at the fixed ends of the apparatus. Evidently the number of vortices which constitute the primary flow is dependent on the overall length of the flow domain. Therefore it is of interest to study how one primary flow exchanges its priority with the next one as the length is varied. In this study we have concentrated on the changeover between four and six vortices. The results are shown in figure 4 and are described below.

For values of $\Gamma \leqslant 5.32$ the four-cell state is primary while for $\Gamma \geqslant 6.70$ the six-cell state becomes so. The lengths between these values constitute the primary-flow exchange region. The results shown in figure 4 are very like those obtained for the four–six exchange in the Taylor–Couette system (see for example figure 2 of Mullin 1982). In the present study, however, the upper locus of critical points BC was

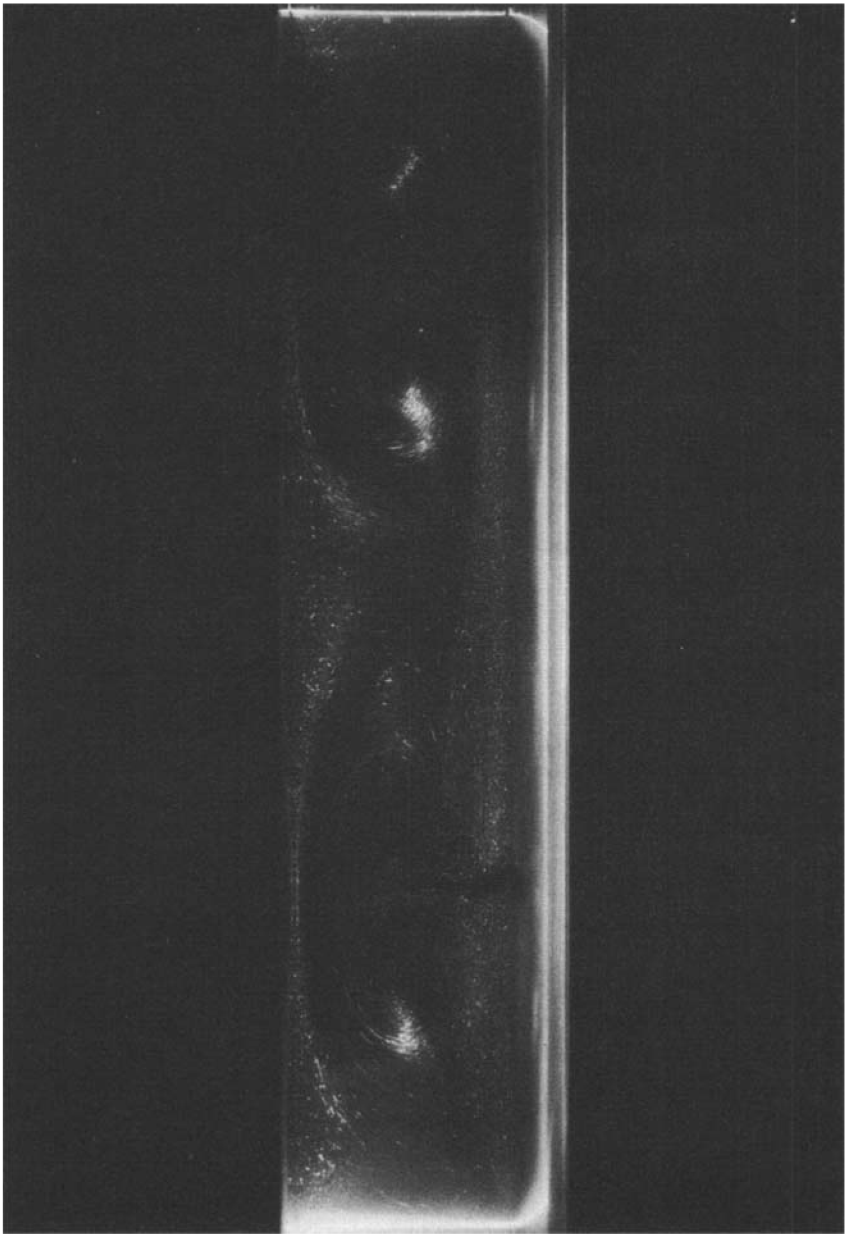
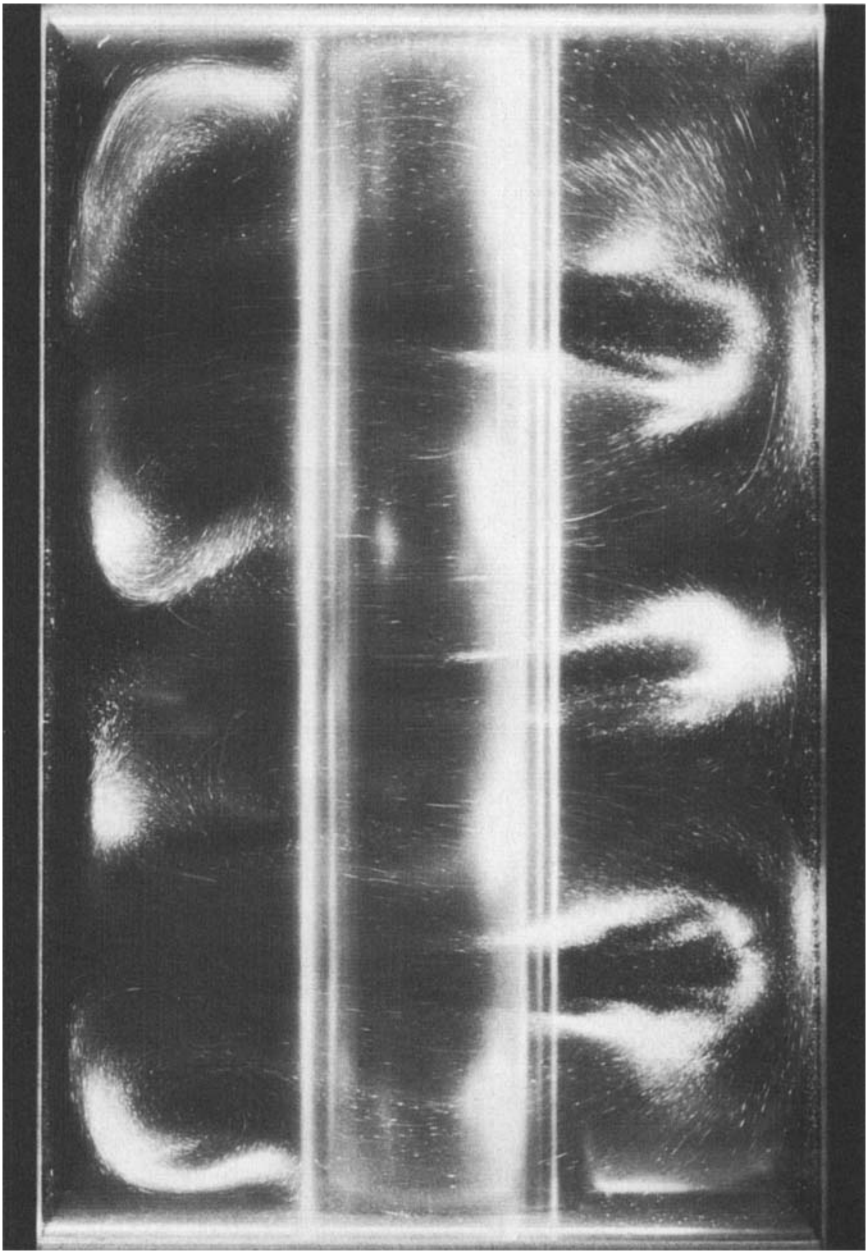


FIGURE 5. Asymmetric four-cell state with larger bottom pair of cells.

unobtainable and for this reason it is indicated by a dashed line. Normally BC would have been obtained by slowly increasing R from small values, being the locus of critical values of R at which a definite six-cell state suddenly appears. For the present geometry other events took precedence at lower values of R and these are described below.

Increasing R from small values we observed the smooth development of a four-cell state which becomes asymmetric and then time dependent. The asymmetric flow is shown in figure 5, where the bottom vortex pair is larger than the top. The alternative

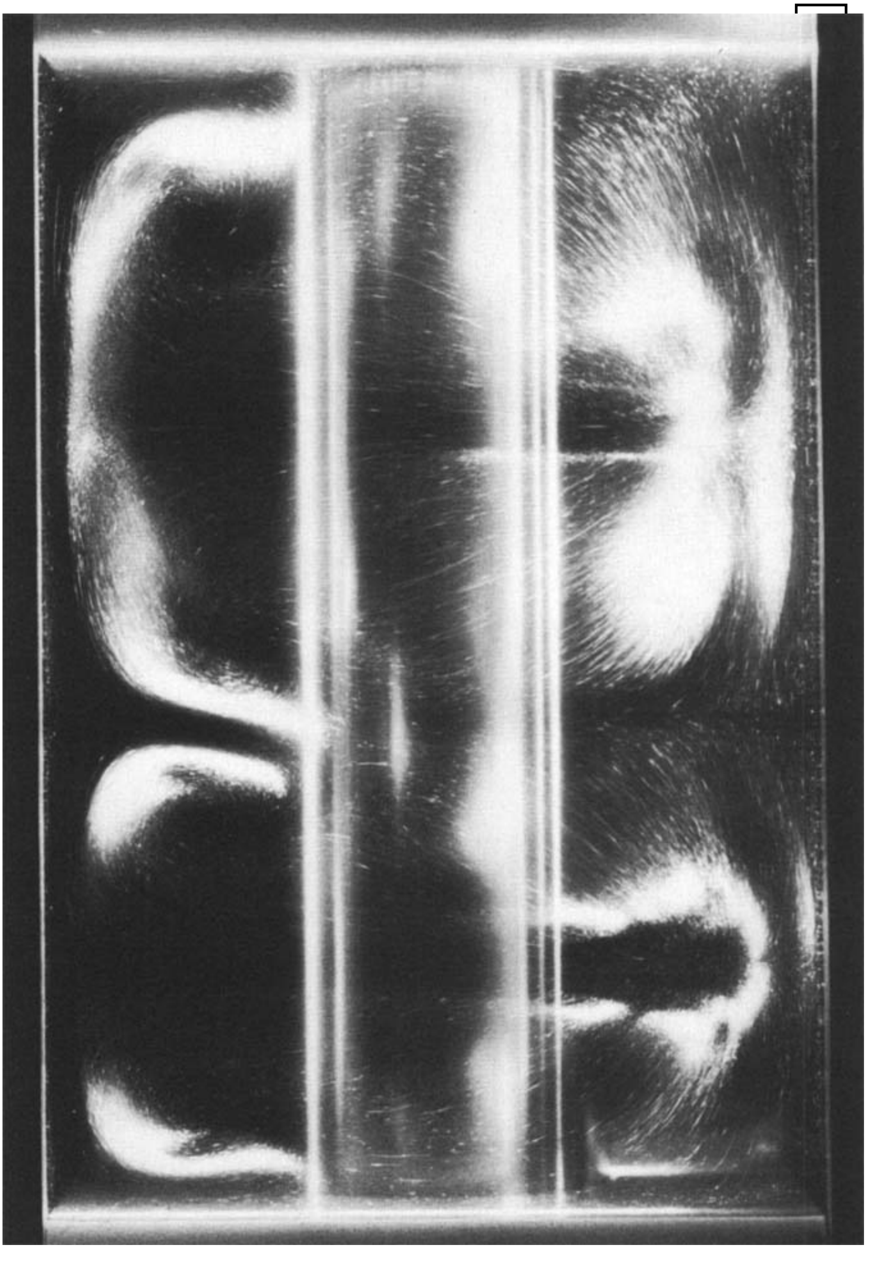


(*a*)

FIGURE 6. For caption see next page.

case of a larger top-cell pair was also observed but was less often found. The difference in cell size at the onset of asymmetry is small and difficult to detect, but it becomes more apparent with increase in R until the appearance of time dependence.

The time-dependent motion is characterized by the formation of a new vortex pair between existing ones followed by a collapse back to the asymmetric four-cell state. The photographs of figure 6 show respectively the formation of the additional vortex pair and the flow a few seconds after the collapse. The collapse begins with the

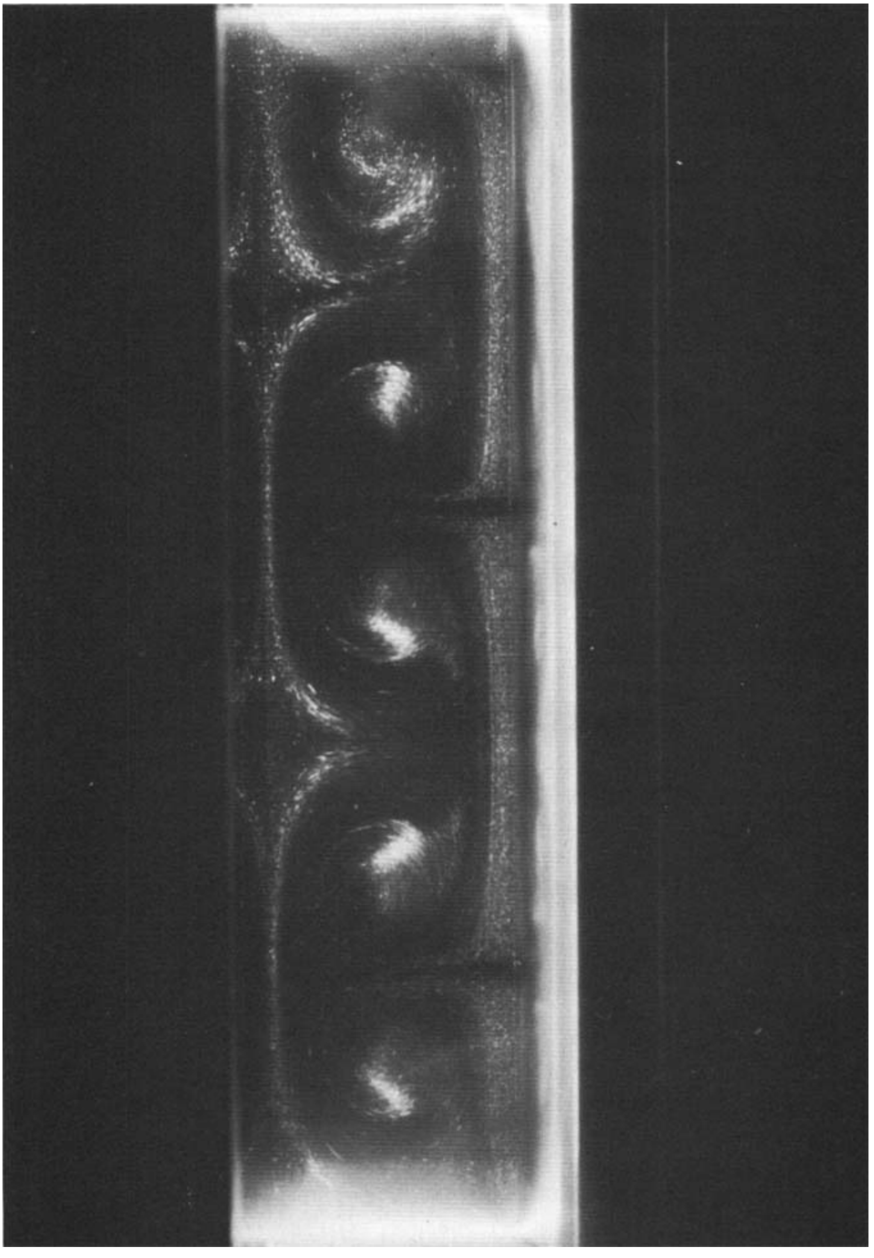


(*b*)

FIGURE 6. Periodic jumping between the four- and six-cell states: (*a*) formation of an additional vortex pair between the two existing pairs; (*b*) a few seconds after the collapse back to the asymmetric four state. In both cases $R = 131$ and $\Gamma = 6.26$.

attraction of the outward flow of two neighbouring vortex pairs until only the inward pair remains. During later experiments, when we used laser-Doppler velocimetry, we confirmed our earlier observation that this process is periodic with a period of around 30 s.

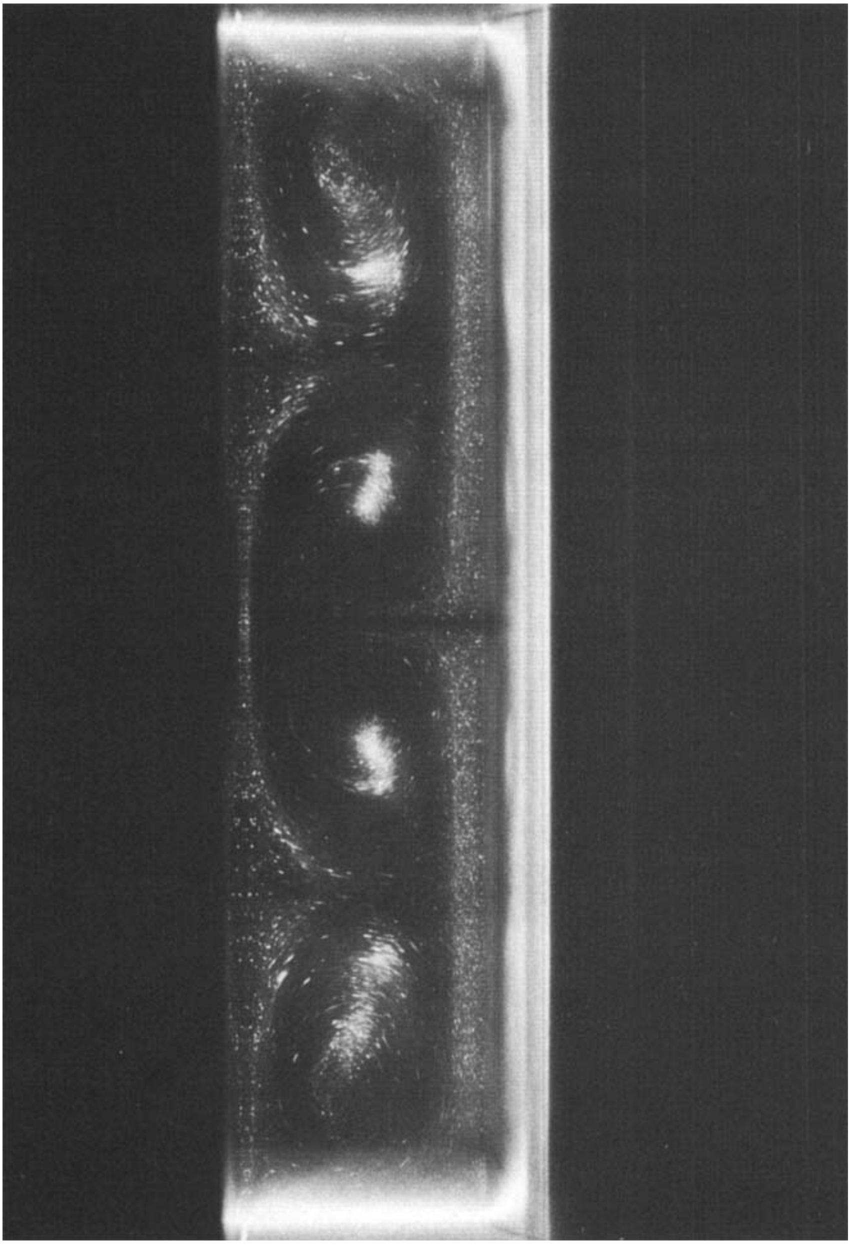
For smaller values of Γ there is a distinct difference between the onset of



(*a*)

FIGURE 7. For caption see next page.

asymmetry, EE′ in figure 4, and the onset of time-dependent flow given by BF. However, for greater aspect ratios both appear simultaneously when E′B is crossed by increase of R. In the range of aspect ratio below A but above E the asymmetry is less than 1 mm and disappears smoothly with increase in R, which suggests that EF may be linked. But, repeatable measurements of the return to symmetry could not be obtained. Below E there is a continuous evolution of the four-cell state by increase of R.



(b)

FIGURE 7. (a) Anomalous five-cell state with the additional cell at the top. (b) Anomalous four-cell state. In both cases $R = 131$ and $\Gamma = 5$.

The loci of stability limits for the secondary modes, shown in figure 4, are readily found by our standard techniques. AB is the locus of the limits of stability for the six-cell mode and CD the corresponding one for the four-cell state. In both cases the secondary mode is formed by switching on to a value of R well above the stability limit. R is then gradually reduced and the limit of stability is taken to be the critical value of R where the mode is observed to collapse catastrophically. The steady four-

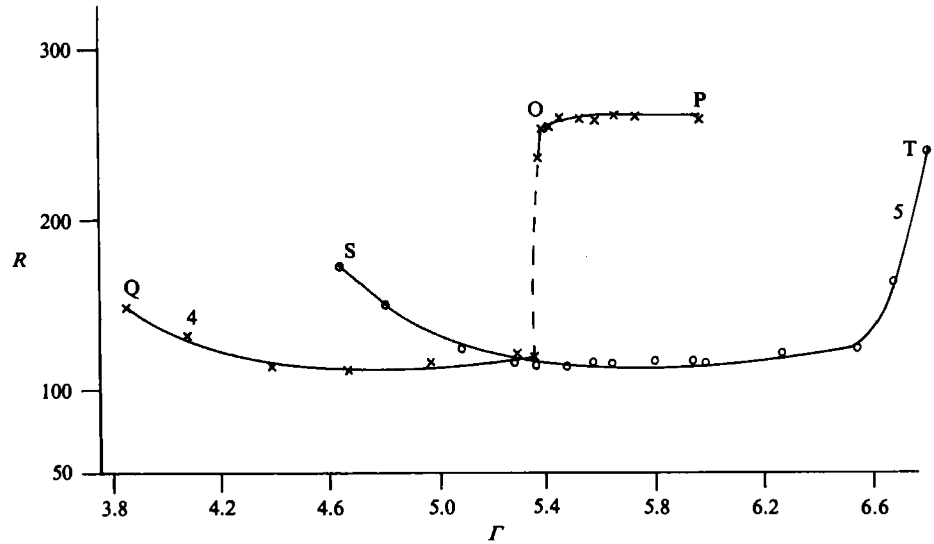


FIGURE 8. Loci of the limits of stability of the anomalous four- and five-cell states as a function of Γ. ST is the locus of limits of stability for the five-cell flow and QOP that for the four cell. Both are found by reducing R.

cell state collapses to the steady six-cell state while the steady six-cell mode generally collapses to the time-dependent state discussed above.

3.2. *Anomalous modes*

As in the case of the Taylor–Couette model steady flows may be created where the spiralling motion of the end cell is such that outward flow may be found at either one or both fixed ends. These are the so-called anomalous modes and were investigated in detail by Benjamin (1978*b*) and by Benjamin & Mullin (1981) for Taylor–Couette flow. They are producible in practice by sudden acceleration of the inner cylinder to a speed approximately three times that for the first appearance of cells. Here we have concentrated our study on the behaviour of the anomalous four-cell and five-cell flows as a function of R and Γ. Photographs of the flows are shown in figure 7 and the results of the measurements are given in figure 8. According to the symmetric boundary conditions applicable, the state with an odd number of cells may have the extra vortex at either the top or the bottom.

The limits of stability of the anomalous modes are found by producing the mode at high values of R and then gradually reducing R until the mode collapses catastrophically at a well-defined critical speed.

The locus of stability limits ST for the five-cell state is qualitatively the same as that found for Taylor–Couette flow. However, the respective curve for the anomalous four-cell state is markedly different for large values of Γ. Here we observed a similar type of asymmetry and time-dependent flow as had been found in the four–six exchange. OP is the locus of critical speeds at which the steady anomalous four-cell mode becomes time dependent when R is reduced. This time dependence is similar to that found for the normal four-cell state and no repeatable critical speeds were found for collapse into another steady state. It is interesting to note that the length above which the time-dependent anomalous four-cell state was found is the same, to within experimental accuracy, as that for the onset of asymmetry in the normal four-cell mode.

The occurrence of asymmetric flows and the associated time dependence at first suggested deficiencies in the construction of the apparatus. The speed of the inner cylinder was carefully monitored, the dimension of the cylinders was checked for accuracy and the experiment was repeated with and without the bottom PTFE collar. Further investigations into the asymmetry included attaching a thin PTFE ring to the inner rotating cylinder midway along its length to try to force a symmetric flow. However, the device disrupted the entire flow and steady states could not be found. Another attempt consisted of a study of the exchange between two-cell and three-cell states with a free top surface. Again the familiar periodic jumping between cellular modes was found. Later visualization experiments with a larger apparatus of similar radius ratio also exhibited asymmetric states and periodic jumping in cell number in the primary flow exchange region.

4. Conclusion

The properties of steady cellular flows between a rotating circular and a stationary square outer cylinder have been investigated. Both the primary-flow exchange process and the properties of the anomalous modes are qualitatively the same as those observed in standard Taylor–Couette flow. The radical change in geometry has not altered the characteristic features of the stability properties disposed among observed steady motions and they are in accord with the general theoretical framework proposed by Benjamin.

The principal differences between the behaviour of the flows here observed and that of those in the Taylor–Couette model can be accounted for by the presence of asymmetric states and their exceptional tendency to time dependence. The formation of an asymmetric flow through a symmetric bifurcation has been observed previously in the Taylor–Couette system only when the aspect ratio is very small (Benjamin & Mullin 1981). In the present study, on the other hand, asymmetric states were found to be prevalent over a wide range of lengths. Moreover, attempts to suppress these complicating effects by use of a free surface and by introduction of boundary perturbations were not successful.

The periodic jumping between the four-cell and six-cell states may be due to the presence of a codimension-2 bifurcation, as expounded by Guckenheimer & Holmes (1983). The midplane symmetry of the observed flow is lost upon crossing the line E′E of figure 4 by increasing R, which indicates the presence of a symmetry-breaking bifurcation of the type discussed by Golubitsky & Schaeffer (1979). If the line of limit points of the fold BC (not detected in the experiments) and the line of symmetry-breaking bifurcation points intersect, then a codimension-2 bifurcation arises at the point of intersection. Guckenheimer & Holmes argue that a line of Hopf bifurcation points will emerge from such a point. A characteristic feature of the oscillation so arising is its relatively long period. These features are all found in the present system, which finding leads us to conjecture the presence of a codimension-2 bifurcation.

The authors would like to thank Dr G. Pfister of the University of Kiel for help in some of the experiments and Professor T. B. Benjamin for constructive comments upon an earlier version of this paper. T.M. acknowledges the support of the S.E.R.C.

REFERENCES

BENJAMIN, T. B. 1978*a* Bifurcation phenomena in steady flows of a viscous fluid. I. Theory. *Proc. R. Soc. Lond.* A **359**, 1–26.

BENJAMIN, T. B. 1978*b* Bifurcation phenomena in steady flows of a viscous fluid. II. Experiment. *Proc. R. Soc. Lond.* A **359**, 27–43.

BENJAMIN, T. B. & MULLIN, T. 1981 Anomalous modes in the Taylor experiment. *Proc. R. Soc. Lond.* A **377**, 221–249.

CLIFFE, K. A. 1983 Numerical calculations of two-cell and single-cell Taylor flows. *J. Fluid Mech.* **135**, 219–233.

CLIFFE, K. A. & MULLIN, T. 1985 A numerical and experimental study of anomalous modes in the Taylor experiment. *J. Fluid Mech.* **153**, 243–258.

GOLUBITSKY, M. & SCHAEFFER, D. 1979 Imperfect bifurcation in the presence of symmetry. *Communs Math. Phys.* **67**, 205–232.

GUCKENHEIMER, J. & HOLMES, P. 1983 *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*, chap. 7. Springer.

HALL, P. 1982 Centrifugal instabilities of circumferential flows in finite cylinders: the wide gap problem. *Proc. R. Soc. Lond.* A **384**, 359–379.

LEWIS, E. 1979 Steady flow between a rotating circular cylinder and a fixed square cylinder. *J. Fluid Mech.* **95**, 497–513.

MULLIN, T. 1982 Mutations of steady cellular flows in the Taylor experiment. *J. Fluid Mech.* **121**, 207–218.

MULLIN, T., LORENZEN, A. & PFISTER, G. 1983 Transition to turbulence in a non-standard rotating flow. *Phys. Lett.* **96A**, 236–238.

MULLIN, T., PFISTER, G. & LORENZEN, A. 1982 New observations on hysteresis effects in Taylor–Couette flow. *Phys. Fluids* **25**, 1134–7.

SCHAEFFER, D. G. 1980 Analysis of a model in the Taylor problem. *Math. Proc. Camb. Phil. Soc.* **87**, 307–337.

SNYDER, H. A. 1968 Experiments on rotating flows between non-circular cylinders. *Phys. Fluids* **11**, 1606–1611.